4 Chapter 4 Minimization and Optimization

4.1 Introduction

To perform minimization and optimization, the program module sequence of Figure 4-1 applies.

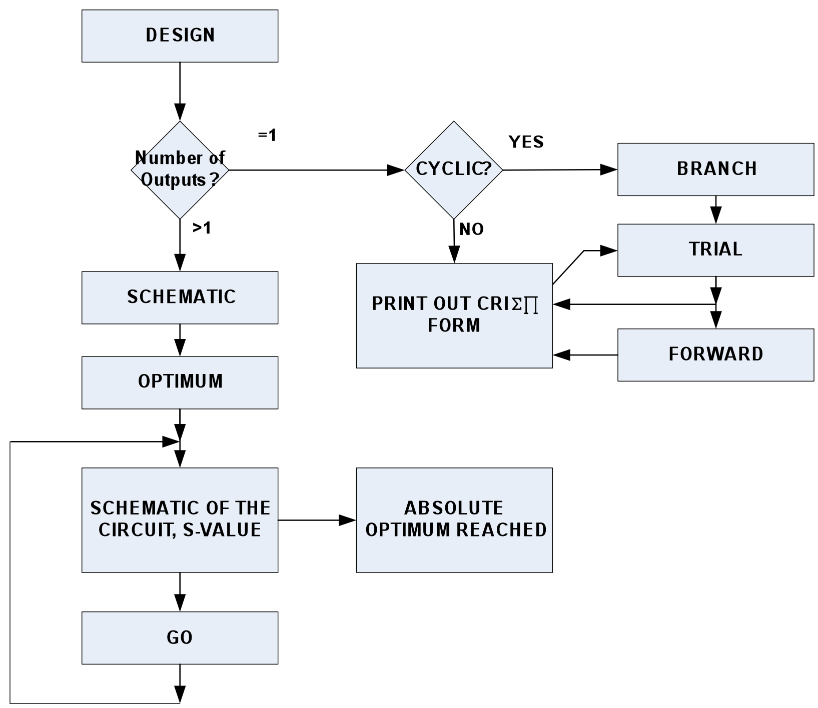

Figure 4-1 Program Module Sequence for Minimization and Optimization

To begin, we type:

)*LOAD* <library number> *OPTIMA*

DESIGN

The interaction is demonstrated in the following examples.

4.2 Examples

Example 4.2.1. Select a Boolean function of 5 variables at random and minimize it $\Sigma\prod$ form. To develop such a function, we call:

```
      EXAMPLE
TYPE NUMBER OF VARIABLES:
⎕:
      5
MINS→→→ 0  2  5  7  12  13  16  19  20  22  26  27  29  30
DONTS→→→3  8  11  15  23  25
```

The function is specified by decimal equivalents. APL vectors **MINS** and **DONTS** are ***mutually exclusive***, and they can be used in the program that follows. (MINS represent points where the function is specified as true).

We are ready to call:

```
      DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLEAN FUNCTIONS:
⎕:
      5
THEIR SYMBOLS:  ABCDE
SYMBOLISN EXPLANTATION:
OK   MEANS: NO MISTAKE
FLT  MEANS: FAULTY TYPING, REQUEST FOR RETYPING
ADD  MEANS: REQUEST FOR ADDITIONAL DATA INSERTION
```

This explanatory text is intended for students. It is easy to cancel it in the program. In the examples that follow, we will not repeat the printout of explanatory comments. Note that the number of outputs is equal to 1 when we minimize the $\Sigma\prod$ form of a given function.

```
TYPE THE NUMBER OF OUTPUTS. (BY TYPING:  1 THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
FUNCTION.)
⎕:
      1
TYPE-IN DECIMAL EQUIVALENTS OF TRUE MINTERMS
OR THE SYMBOL OF THE CORRESPONDING VECTOR.
⎕:
      MINS
TYPE: OK OR FLT OR ADD:
⎕:
      OK
TYPE IN UNSPECIFIED MINTERMS (IF NONE, TYPE: NONE).
⎕:
      DONTS
TYPE: OK OR FLT OR ADD:
⎕:
      OK
WEIGHT TABLE:

W = 2 FOR MINS:  2  5  12  16  20  26  29  30
W = 3 FOR MINS:  0  19  22
W = 4 FOR MINS:  7  13  27
```

Marquand charts will be used.

Consult the weight table to estimate the difficulty of the problem. For a function of more than 6 variables which happens to have only a small count of low weight minterms (W = 0, 1, 2, 3), it is advisable to begin with the lowest weight value found in the table as a limit set for the execution. This limit can be increased later one by one when it becomes clear that the generation of new terms is adequate and the corresponding execution time is acceptable.

```
SATE OF THE CRITICAL SET:
SET A LIMIT FOR W OR TERMINATE BY TYPING:  0
⎕:
```


The weight ***W*** has a very simple meaning. It is equivalent to the ***count*** of those non-zeros of the given function, which are at a ***unit logical distance*** from a minterm implicant.

For instance, W = 2 for MIN identified with 5 means: There are exactly two non-zeros of the function at unit logical distance from the MINterm 5 = $(00101)_2 \equiv \underline{ED}B\underline{C}A$:

7 = $(00111)_2$ and 13 = $(01101)_2$, (both belonging to MINS).

W=4 for the MINterm 7 = $(00111)_2 \equiv \underline{ED}BCA$ means that there are four non-zeros: 5 = $(00101)_2$, a MIN, and 3 = $(00011)_2$, 15 = $(01111)_2$, and 23 = $(10111)_2$, all members of DONTS.

For this example, the limit for W is to be set to 4, so we type:

```
PRESENT STATE
1   0   1   *   0   1   0   1
*   0   0   *   1   1   0   *
1   0   0   1   1   0   1   *
0   *   1   1   0   1   1   0
```

N–MINIMAL FORM:

$\underline{A}\ \underline{CDE}$ 2

$A\ C\ \underline{E}$ 5

$\underline{AB}\ D\underline{E}$ 16

$A\underline{B}CD$ 29

$AB\underline{C}$ 19

$\underline{A}B\ DE$ 26

$ABC\ E$ 22

The N-minimal $\Sigma\prod$ form of the given incompletely specified function is:

$$\underline{ACDE} + AC\underline{E} + \underline{AB}D\underline{E} + \underline{ABD}E + A\underline{B}CD + AB\underline{C} + \underline{A}BDE + \underline{A}BCE$$

{ 2 , 5 , 12 , 16 , 29 , 19 , 26 , 22 }

The corresponding ***critical set*** is presented below the minimal form. Any pair of MINS taken from this set has components that are ***mutually term exclusive*** (MTE). This means that no implicant T => f of the given function exists that covers both components of any pair.

(For functions with a cycle, i.e., ***cyclic functions***, the printed set is not always ***"critical"***!) The element of the critical set printed here below a term is covered by that term ***exclusively***.

For instance, MINterm 19 is covered by $AB\underline{C}$ and not by any other term of the $\Sigma\prod$ form.

Exercise: Sketch the two-level NAND network defined by the $\Sigma\prod$ form.

Example 4.2.2. Design a two-level NOR network generating the function defined in **Example 4.2.1**.

Procedure: Develop an N-minimal $\Sigma\prod$ form for the complement $\underline{\mathbf{f}}$ of that function and apply DeMorgan's rules to obtain the minimal $\prod\Sigma$ form of **f**. Note: DONTS remain unchanged and the $\underline{\mathbf{f}}$ must be true where **f** was specified as false.

MINS1←1   4   6   9   10   14   17   18   21   24   28   31

We again call DESIGN (the explanatory texts are not reproduced here):

```
      DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLEAN FUNCTIONS:
⎕:
      5
THEIR SYMBOLS:   ABCDE

TYPE THE NUMBER OF OUTPUTS. (BY TYPING:  1  THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
FUNCTION.)
⎕:
      1
TYPE-IN DECIMAL EQUIVALENTS OF TRUE MINTERMS
OR THE SYMBOL OF THE CORRESPONDING VECTOR.
⎕:
      MINS1
TYPE: OK OR FLT OR ADD:
⎕:
      OK
TYPE IN UNSPECIFIED MINTERMS (IF NONE, TYPE: NONE).
⎕:
      DONTS
TYPE: OK OR FLT OR ADD:
⎕:
      OK
WEIGHT TABLE:

W = 0     FOR MINS:     18
W = 1     FOR MINS:     4  28
W = 2     FOR MINS:     6  21  31
W = 3     FOR MINS:     1  101114  17  24
W = 4     FOR MINS      9

STATE OF THE CRITICAL SET:

SET A LIMIT FOR W OR TERMINATE BY TYPING:  0
⎕:
      4
PRESENT STATE
0  1  0  *  1  0  1  0
*  1  1  *  0  0  1  *
0  1  1  0  0  1  0  *
1  *  0  0  1  0  0  1

N-MINIMAL FORM:
```

$\underline{A}B\underline{CD}E$	18
$A\ C\underline{DE}$	4
$\underline{AB}\ DE$	28
$ABCD$	31
$A\underline{BC}$	1
$B\ D\underline{E}$	10

The N-minimal ΣΠ form of $\underline{f}$ is

$\underline{A}B\underline{CDE} + \underline{A}C\underline{DE} + \underline{AB}DE + A\underline{BD}E + ABCD + A\underline{BC} + BD\underline{E}$

(1 28, 21, 31, 1, 10)

and the ΠΣ of f (by DeMorgan's rules) is

$f = (A+\underline{B}+C+D+\underline{E})(A+\underline{C}+D+E)(A+B+\underline{D}+\underline{E})(\underline{A}+B+D+\underline{E})(\underline{A}+\underline{B}+\underline{C}+\underline{D})(\underline{A}+B+C)(\underline{B}+\underline{D}+E)$

Exercise: Sketch the corresponding two-level NOR circuit. Discuss which of the two circuits (two-level NAND or two-level NOR) is more practical.

Example 4.2.3. Design a full adder as a combinational circuit with multiple outputs (see Figure 1-3). The output functions are:

(Z 1) ≡ {1, 2, 4, 7}

(Z 2) ≡ {3, 5, 6, 7}

The procedure again starts with:

```
      DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLEAN FUNCTIONS:
⎕:
      3
THEIR SYMBOLS:  ABC

TYPE THE NUMBER OF OUTPUTS. (BY TYPING: 1 THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
FUNCTION.)
⎕:
      2
DEFINE FUNCTIONS BY DECIMAL EQUIVALENTS.  TYPE: NONE FOR
EMPTY SETS.
FUNCTION LABELED 1 IS TRUE AT:
⎕:
      1  2  4  7

TYPE: OK OR FLT OR ADD:
⎕:
      OK

PRESENT STATE:
* * * * * * * *              |
0 1 1 0 1 0 0 1              |
0 0 0 1 0 1 1 1              |      mosaic function
0 0 0 0 0 0 0 0              |
```

N-MIMIMAL FORM

$A\underline{BC}\ \underline{E}$ 9

$\underline{A}B\underline{C}\ \underline{E}$ 10

$\underline{AB}C\ \underline{E}$ 12

ABC 15

$AB\ \underline{D}$ 19

$A\ C\underline{D}$ 21

$BC\underline{D}$ 22

You may call SCHEMATIC. The resulting graph describes the circuit under the following rules:

- Horizontally Aligned Quads (----⎕----) represent the inputs of an AND-gate
- Horizontally aligned circles (---o----) represent the input of an OR-gate whose output variable is designated by a literal printed above the circular marker (o).

The purpose of the procedure up to now was to develop a critical set of the Mosaic Function. This critical set is a very important factor of the procedure that develops the two-level circuit with the minimum of gates and, within this condition, with the minimum number of gate inputs.

The development of the circuit from the S-minimal form is understood after a comparison between the N-minimal form (see above) and the circuit printed by the terminal:

```
   SCHEMATIC
FIRST APPROXIMATION OF THE OPTIMAL NETWORK"
 A     A     B     B    C    C    D      E
-|-----▯-----▯-----|----▯----|----o------|---
-▯-----|-----|-----▯----▯----|----o------|---
-▯-----|-----▯-----|----|----▯----o------|---
-|-----▯-----|-----▯----|----▯----o------o---
-|-----▯-----|-----▯----|----|----|------o---
-|-----▯-----|-----|----|----▯----|------o---
-|-----|-----|-----▯----|----▯----|------o---

S-VALUE OF THE CIRCUIT:  12
CALL OPTIMUM

   OPTIMUM
S-VALUE HAS BEEN REDUCED TO:  11
ABC E          9
ABC E          10
ABC E          12
ABC E          15
AB D           19
A CD           21
 BCD           22

AND THE CIRCUIT TO:
A     A     B     B    C    C    D      E
-|-----▯-----▯-----|----▯----|----o------|---
-▯-----|-----|-----▯----▯----|----o------|---
-▯-----|-----▯-----|----|----▯----o------|---
-|-----▯-----|-----▯----|----▯----o------|---
-|-----▯-----|-----▯----|----|----|------o---
-|-----▯-----|-----|----|----▯----|------o---
-|-----|-----|-----▯----|----▯----|------o---

S-VALUE OF THE CIRCUIT:  11
TYPE: GO  IF YOU WANT TO CONTINUE.  IF NOT, TYPE: STOP.
▯:
   GO
ABSOLUTE OPTIMUM REACHED.
```

By calling OPTIMUM, the S-minimization is started. To prevent undesirable cost of execution time, the circuit is reprinted each time the S-value drops by one unit. Calling G0 can be continued as long as desirable.

When the terminal prints ABSOLUTE MAXIMUM REACHED, the provably best solution has been reached. (Exception: When one or more of the functions has a single literal implicant such as A, <u>A</u>, B, … , the problem must be solved twice, once in the normal way described previously and the second time with A, <u>A</u>, B, … used as the output OR-gate input and with single literal implicants.)

Example 4.2.4. Design a full adder with outputs (Z 1), (<u>Z</u> 2) (the second output is complimented!). This time, we reproduce the printout without any comments:

```
   DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLEAN FUNCTIONS:
▯:
   3
THEIR SYMBOLS:  ABC

TYPE THE NUMBER OF OUTPUTS. (BY TYPING:  1 THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
```



```
FUNCTION.)
⎕:
    2
DEFINE FUNCTIONS BY DECIMAL EQUIVALENTS. TYPE:  NONE FOR
EMPTY SETS.
FUNCTION LABELED 1 IS TRUE AT:
⎕:
    1 2 4 7           ←----Function “D”
TYPE: OK OR FLT OR ADD:
⎕:
    OK
FUNCTION NUMBER 1 IS UNSPECIFIED AT:
⎕:
    NONE
TYPE: OK OR FLT OR ADD:
⎕:
    OK
FUNCTION LABELED 2 IS TRUE AT:
TYPE: OK OR FLT OR ADD:
⎕:
    0 1 2 4             ←----Function “E”

TYPE: OK OR FLT OR ADD:
⎕:
    OK
FUNCTION NUMBER 2 IS UNSPECIFIED AT:
⎕:
    NONE
TYPE: OK OR FLT OR ADD:
⎕:
    OK
WEIGHT TABLE:
W = 1      FOR MINS:     15
W = 2      FOR MINS:     9 10 12
W = 3      FOR MINS:     17 18 20
W = 4      FOR MINS:     16

STATE OF THE CRITICAL SET:

SET A LIMIT FOR W OR TERMINATE BY TYPING: 0
⎕:
    4
PRESENT STATE:
```

** * * * * * * *   ←- $\underline{E}$ $\underline{D}$*
0 1 1 0 1 0 0 1   ←- $\underline{E}$ D
1 1 1 0 1 0 0 0   ←- E $\underline{D}$
*0 * * 0 * 0 0 0   ←- E D*

Note: The Mosaic function composed for this purpose has:

- The row $\underline{E}\underline{D}$ filled with DON’T CAREs (always).
- The row $\underline{E}$D filled with the function D
- The row E$\underline{D}$ filled with the function E
- The row ED filled with DON’T CAREs wherever the product ED is NOT zero.

In general, for more outputs (for instance, four: D, E, F, and G), the functions are filled only into rows with a single non-complemented variable:

Contains: $\dfrac{G\underline{FED}}{G}$ $\dfrac{\underline{G}F\underline{ED}}{F}$ $\dfrac{\underline{GF}E\underline{D}}{E}$ $\dfrac{\underline{GFE}D}{D}$

Any other row contains DON'T CAREs only. For instance, the row $G\underline{F}E\underline{D}$ contains a DON'T CARE whenever the product of functions G and E is not equal to zero.

N–MINIMAL FORM:

$ABC\ \underline{E}$	15
$A\underline{BC}$	9
$\underline{A}B\underline{C}$	10
$A\underline{B}C$	12
$\underline{AB}\ \underline{D}$	16

You may call SCHEMATIC. The resulting graph describes the circuit under the following rules:

- Horizontally Aligned Quads (----▯----) represent the inputs of an AND-gate
- Horizontally aligned circles (---- o----) represent the input of an OR-gate whose output variable is designated by a literal printed above the circular marker (o).

SCHEMATIC

FIRST APPROXIMATION OF THE OPTIMAL NETWORK:

$\underline{A}$ A $\underline{B}$ B $\underline{C}$ C D E

```
-|-----▯-----|----▯-----|-----▯---------O-------|--
-|-----▯-----▯----|-----|-----|---------O-------O--
-▯-----|-----|----▯-----|-----|---------O-------O--
-▯-----|-----▯----|-----|-----▯---------O-------O--
-▯-----|-----▯----|-----|-----|---------|-------O--
```

S–VALUE OF THE CIRCUIT: 12

CALL OPTIMUM.

OPTIMUM

S–VALUE HAS BEEN REDUCED TO: 11

$ABC\ \underline{E}$	15
$A\underline{BC}$	9
$\underline{A}B\underline{C}$	10
$\underline{AB}C\ E$	12
$\underline{AB}\ \underline{D}$	16

AND THE CIRCUIT TO:

$\underline{A}$ A $\underline{B}$ B $\underline{C}$ C D E

```
-|-----▯-----|----▯-----|-----▯---------O-------|--
-|-----▯-----▯----|-----▯-----|---------O-------O--
-▯-----|-----|----▯-----▯-----|---------O-------O--
-▯-----|-----▯----|-----|-----▯---------O-------|--
-▯-----|-----▯----|-----|-----|---------|-------O--
```

S–VALUE OF THE CIRCUIT: 11

TYPE: GO IF YOU WANT TO CONTINUE. IF NOT, TYPE: STOP.

▯:

GO

ABSOLUTE OPTIMUM REACHED.

The resulting circuit (with ($\underline{Z}$ 2) as output) is simpler than the circuit obtained in Example 4.2.3. It is used in integrated circuits.

Example 4.2.5. The Boolean function: MINS = 1, 2, 7; DON'Ts – 0, 3, 5, 6, is at the same time CYCLIC and ABNORMAL. Any critical set contains one element at the maximum, because the elements of pairs: (1, 2), (1, 7), (2, 7) are MUTUALLY TERM INCLUSIVE (MTI).

The maximal number of elements in a critical set is M = 1. The minimal number of terms a $\Sigma\Pi$ -from, however, is N = 2, so N > M, a property called ABNORMAL. We are interested in the result of the minimization procedure DESIGN applied to that function.

```
    DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLENA FUNCTIONS:
⎕:
    3
THEIR SYMBOLS:  ABC
TYPE THE NUMBER OF OUTPUTS. (BY TYPING: 1 THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
FUNCTION.)
⎕:
    1
TYPE-IN DECIMAL EQUIVALENTS OF TRUE MINTERMS
OR THE SYMBOL OF THE CORRESPONDING VECTOR.
⎕:
    1 2 7
TYPE: OK OR FLT OR ADD:
⎕:
    OK
TYPE IN UNSPECIFIED MINTERMS (IF NONE, TYPE : NONE).
⎕:
    OK
WEIGHT TABLE:

W = 3     FOR MINS:     1 2 7

STATE OF THE CRITICAL SET:
SET A LIMIT FOR W OR TERMINATE BY TYPING: 0
⎕:
    3
PRESENT STATE:
* 1 1 *
0 * * 1

CYCLIC. PARTIAL FORM:
RIGHT-HAND-SIDE BORDER INTEGERS
BELONG TO A CRITICAL SET.
RESIDUAL FUNCTION:
* 1 1 *
0 * * 1

TO CONTINUE CALL BRANCH
```

We were informed that the case is CYCLIC and that the minimization process would not be started (PARTIAL FORM is empty). The residual function is identical to the function we started with. Branching is done in the usual way:

```
    BRANCH

BRANCHING POINT CHOSEN IS: 1
```

```
TABLE OF BRANCHING TERMS:
A        1
  C      2
INTEGERS AT THE RIGHT-HAND-SIDE ARE SO CALLED: ROW NUMBERS.
CALL TRIAL.

    TRIAL
SELECT BRANCHING TERM BY TYPING ITS ROW NUMBER.
⎕:
    1
BRANCHING TERM:
A      1

COMPLETE COVERAGE. RESULTING FORM:
A      1
 B     2

RIGHT-HAND-SIDE BORDERS INTEGERS
BELONG TO A CRITICAL SET.
LITERALS TOTAL: 2
MAY CALL TRIAL AGAIN.
```

Because the case is cyclic, the integers printed on the right-hand side border do not represent a critical set! The N-minimal form, however, is correct. By calling TRIAL again, we can form all branches and compare them:

```
    TRIAL
SELECT BRANCHING TERM BY TYPING ITS ROW NUMBER.
⎕:
    2
BRANCHING TERM:
  C    1
COMPLETE COVERAGE. RESULTING FORM:
  C    1
A      7

RIGHT-HAND-SIDE BORDERS INTEGERS
BELONG TO A CRITICAL SET.
LITERALS TOTAL: 2
MAY CALL TRIAL AGAIN.
```

Remarks: The cycle is ODD: A, B, $\underline{C}$ are three existing “primimplicants” forming that cycle. The function is abnormal (M = 1, N = 2). The N-minimal forms A + B, A + $\underline{C}$ are correct.

Example 4.2.6. The completely specified function: MINS = 0, 1, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14 is highly cyclic but normal. Minimize with DESIGN:

```
    DESIGN
NUMBER OF INDEPENDENT VARIABLES OF GIVEN BOOLENA FUNCTIONS:
⎕:
    4
THEIR SYMBOLS:  ABCD
TYPE THE NUMBER OF OUTPUTS. (BY TYPING: 1 THE PROCEDURE
IS REDUCED TO AN N-MINIMIZATION OF A SINGLE GIVEN
```



```
FUNCTION.)
⎕:
    1
TYPE-IN DECIMAL EQUIVALENTS OF TRUE MINTERMS
OR THE SYMBOL OF THE CORRESPONDING VECTOR.
⎕:
    0 1 3 4 6 7 9 10 11 12 13 14
TYPE: OK OR FLT OR ADD:
⎕:
    OK
TYPE IN UNSPECIFIED MINTERMS (IF NONE, TYPE : NONE).
⎕:
    NONE
TYPE: OK OR FLT OR ADD:
⎕:
    OK

WEIGHT TABLE:

W = 2     FOR MINS:    0 7 10 13
W = 3     FOR MINS:    1 3 4 6 9 11 12 14

STATE OF THE CRITICAL SET:
SET A LIMIT FOR W OR TERMINATE BY TYPING: 0
⎕:
    3
PRESENT STATE:
1 1 0 1
1 0 1 1
0 1 1 1
1 1 1 0

CYCLIC. PARTIAL FORM:
RIGHT-HAND-SIDE BORDER INTEGERS
BELONG TO A CRITICAL SET.
RESIDUAL FUNCTION:
1 1 0 1
1 0 1 1
0 1 1 1
1 1 1 0

TO CONTINUE CALL BRANCH

    BRANCH
BRANCHIONG POINT CHOSENIS: 0
TABLE OF BRANCHING TERMS:
```

$\underline{AB}\ \underline{D}$	1
$\underline{BCD}$	2

```
INTEGERS AT THE RIGHT-HAND-SIDE ARE SO CALLED: ROW NUMBERS.
CALL TRIAL.
```

TRIAL
SELECT BRANCHING TERM BY TYPING ITS ROW NUMBER.
⎕:
1
BRANCHING TERM:
$\underline{A}\underline{B}\ \underline{D}$ 0

COMPLETE COVERAGE. RESULTING FORM:

$\underline{A}\underline{B}\ \underline{D}$	0	|	
$A\ \underline{C}$	1	|	
$BC\underline{D}$	7	|	←--- NOT A CRITICAL SET
$\underline{A}B\ D$	10	|	
$\underline{B}CD$	13	|	

RIGHT-HAND-SIDE BORDER INTEGERS
BELONG TO A CRITICAL SET.
LITERALS TOTAL: 14
MAY CALL TIRAL AGAIN.

TRIAL
SELECT BRANCHING TERM BY TYPING ITS ROW NUMBER.
⎕:
2
BRANCHING TERM:
$\underline{B}\underline{C}\underline{D}$ 0

COMPLETE COVERAGE. RESULTING FORM:

$\underline{B}\underline{C}\underline{D}$	0	|	
$A\ \underline{C}$	4	|	
$AB\ \underline{D}$	7	|	←--- CRITICAL SET
$B\underline{C}D$	10	|	
$A\underline{B}\ D$	13	|	

RIGHT-HAND-SIDE BORDER INTEGERS
BELONG TO A CRITICAL SET.
LITERALS TOTAL: 14
MAY CALL TIRAL AGAIN.

Remark: The program OPTIMA is well prepared to handle Boolean Functions with one cycle only. When faced with functions possessing multiple cycles, use the program SYSTEM.